A NEW FILTER DESIGN BY THE POTENTIAL ANALOGUE METHOD

MILTON G. WEBB

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MILTON G. WEBB

This work is accepted as fulfilling

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A NEW FILTER DESIGN

BY THE

POTENTIAL ANALOGUE METHOD

by

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Lieutenant, United States Navy

Submitted in partial fulfillment
of the requirements
for the degree of
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PREFACE

The work upon which this thesis is based was performed at the United States Naval Postgraduate School and at the Stanford Research Institute during the academic year 1953-1954. While studying network synthesis in general, and the potential analogue method in particular, an industrial tour was taken at the Stanford Research Institute. During this tour a need was made known to the writer for the type of filter described in the text. Subsequently the design of this filter type was developed.

Appreciation is extended to Professor Robert Kahal of the United States Naval Postgraduate School and to Mister Donald K. Weaver of the Stanford Research Institute. Professor Kahal inspired the writer's initial interest in the field of network synthesis and Mister Weaver suggested the particular problem about which this thesis centers. Both gave freely of their time and provided guidance throughout this work.

25357

TABLE OF CONTENTS

LIST OF ILLUSTRATIONS

SUMMARY

The focal point of the discussion which follows is the design of a filter type having two salient specifications. The first is that the gain function shall be maximally flat in the pass band and the second is that there shall be equal maxima of a specified value in the stop band. A method, utilizing the potential analogue method, is presented for easily obtaining this type of function. Design procedures are developed for directly obtaining the final gain characteristic from the given requirements prior to performing any of the calculations required in the design of the actual filter. Tabulation is made of certain calculated data which are of interest to the design engineer.

CHAPTER I

INTRODUCTION

1. Types of synthesis problems.

There are two main categories of synthesis problems. These are the design of filters and the design of equalizers. Filters may be classified as (1) notch, (2) peak, (3) band suppression, (4) band pass, and (5) any combination of the preceeding. Other characteristics such as type of feed, impedance matching, et cetera, are dependent on physical network configurations. Equalizers may be of the phase correction or gain correction types or both types may be incorporated together. In recent years the phase shift network has become increasingly prominent. This is distinguishable from the phaso equalizer only in that in the phase shift network the phase characteristic is desired for itself rather than as a corrective measure.

2. The development of modern network synthesis.

Historically, synthesis of electric circuits evolved from circuit analysis. Circuits known to have a particular type of characteristic were analyzed and the circuit constants then adjusted to locate particular values of the known characteristic where desired. An advancement of this scheme was to optimize a somewhat variable characteristic so that it matched as closely as possible the one desired. Modern network synthesis is the reverse of the above procedures. The desired characteristic is obtained or approximated in an algebraic form constrained only by realizability conditions. Then the network is derived from this algebraic function. With this procedure one is not limited by the extent of his experience with various circuits in the design of new ones.

In 1924 R. M. Foster(5) treated two-terminal networks containing only reactances. This may be considered the beginning of modern network synthesis. The problem involving the general two-terminal network was solved by O. Brune(2) in 1931 and its practical use was extended by Bott and Duffin(1) in 1949 since they avoided the ideal transformer in the realization of the general two-terminal network. The prime aim of network synthesis might be considered to be the development of desired transfer functions within the limitations of physical realization of the corresponding network. As used in this sentence, transfer function simply means a functional relation between electrical quantities at one terminal-pair and those at another. For two-terminal networks the two terminal-pairs are the same.

Much has been written on two terminal-pair and n terminal-pair networks. Of greater engineering interest at the present time is the former type, and only this category will be considered hereafter. The general realizability requirements have been obtained. Various writers, too numerous to mention, have found some, or all, of the restrictions imposed on the various transfer functions by specific classes of networks with, or without, other restrictions. Others have developed methods of approximating desired characteristics with realizable transfer functions. Other investigations have dealt with realizing the physical network parameters from the transfer function, and in some cases also prescribing the input and output impedances of the network. Those findings, pertinent to the filter designed in this thesis, will be discussed in succeeding chapters.

CHAPTER II

THE APPROXIMATION PROBLEM

1. General.

There are numerous procedures which may be followed in the process of approximation. Before describing any of these, mention should be made of techniques frequently used in the approximation of filter type characteristics. Usually the low pass characteristic is first sought and afterwards modifications are made transforming the characteristic to a band pass or a high pass type, as may be desired. Moreover, certain quantities are usually normalized (i.e., made equal to 1) in the initial development. In the design of equalizers such techniques are not universally applicable.

One of the oldest approximation procedures is to make a realizable characteristic of a network match a desired characteristic exactly at a finite number of given frequencies. Often this gives quite satisfactory results when the number of given matching frequencies is small and when there is a known type of function whose characteristics resemble those desired. A handicap of this method is that little control is held on the difference between the desired characteristic and that which is approximated at frequencies other than the matching ones.

A refinement of the above is the least squares approximation wherein the integral of the squared difference between the two characteristics over the frequency range of interest is made a minimum. This is a decided improvement, but it still leaves one with considerable doubt as to the maximum difference which occurs.

This situation is corrected by techniques wherein the maximum difference is limited. One of these is the equal ripple technique. In this case all the maxima of difference are made equal to the allowed tolerance. This is frequently described as being the most efficient use of the circuit elements. However, characteristics other than the one, or ones, approximated may become intolerable in actual practice.

Another scheme of approximation is to match the desired characteristic at only one frequency and require the difference function to approach monotonically the tolerance in the band of approximation. The maximally flat characteristic, discussed later, exemplifies this type of approximation.

2. The potential analogue method.

Use is made of the potential analogue of network functions in the approximation of these functions by any of the techniques listed in section 1. Laboratory equipment and techniques may be used to determine experimentally the approximating functions. In addition, one's knowledge of electrostatics and potential theory is available to provide analytical tools and intuitive approaches whether laboratory methods or strictly mathematical techniques are used. Darlington(4) has extensively treated the basis and application of the potential analogue method. The following discussion of the analogy is a limited development which is sufficient for the use made of it in chapter III.

A transfer function is defined as the ratio of the output voltage or current of a network to the input voltage or current as a function of p when the input quantity is of the form, $A\,\epsilon^{pt}$. Most generally the real

frequency behavior, for which $p = j\omega$, is that with which one is concerned. The symbol, $t(p)$, represents a transfer function and may be expressed in the form,

$$t(p) = K \frac{\prod_{i=1}^{m} (p - p_{oi})}{\prod_{j=1}^{n} (p - p_{xj})} .$$

For the transfer function to be realizable with a physical, passive network the following restrictions apply:

(a) $\mathrm{Re}\{p_{xj}\} \leq 0$

(b) $t(p)$ is finite for $p = \infty$ and for $p = 0$.

The latter condition is equivalent to the statement, $m \leq n$. The approximation problem then becomes one of determining the p_{oi} and the p_{xj}, subject to the above restrictions, such that the characteristics of $t(p)$ for $p = j\omega$ are tolerably close to those desired.

Consider the logarithm of the transfer function.

$$\log t = \log K + \log\left[\prod_{i=1}^{m}(p - p_{oi})\right] - \log\left[\prod_{j=1}^{n}(p - p_{xj})\right]$$

$$= \log K + \sum_{i=1}^{m} \log (p - p_{oi}) - \sum_{j=1}^{n} \log (p - p_{xj})$$

$$= \log K + \sum_{i=1}^{m} \log |p - p_{oi}| - \sum_{j=1}^{n} \log |p - p_{xj}|$$

$$+ j\left(\sum_{i=1}^{m} \beta_i - \sum_{j=1}^{n} \beta_j\right)$$

where $$r - r_{oi} = |r - r_{oi}| \epsilon^{j\beta_i}$$

and $$r - r_{xj} = |r - r_{xj}| \epsilon^{j\beta_j}$$

Consider next the potential at a point in a plane at a distance, d, from an infinitely long line charge perpendicular to, and passing through this plane. If the line charge has a linear charge density q, then with appropriately chosen units the potential, V, at the point considered is given by:

$$V = - q \log d + \text{const.}$$

The constant is an arbitrary one depending only upon the level chosen for the reference potential. Since potential is a scalar quantity the potential due to several parallel line charges at a point in a perpendicular plane is given by:

$$V = -\sum_{i=1}^{n} q_i \log d_i + \text{const.}$$

where d_i is the distance in the plane of the point at which V is measured from the i^{th} line charge. If the complex notation is used to represent the coordinates in the plane a complex potential, W, may be defined such that:

$$W = -\sum_{i=1}^{n} q_i \log (z - z_i) + \text{const.}$$

where z represents a location in the plane at which W is considered and z_i represents the point in the plane through which the i^{th} line charge passes. From the above equation is obtained:

$$W = \text{const.} - \sum_{i=1}^{n} q_i \log (z - z_i) - j \sum_{i=1}^{n} q_i \theta_i$$

where $z - z_i = |z - z_i| \epsilon^{j\theta_i}$.

Realizing that $|z - z_i|$ is the distance from the i^{th} line charge to the point it is seen that:

$$W = V - j \sum_{i=1}^{n} q_i \theta_i \quad .$$

All future reference to this potential picture will concern quantities in the plane defined above and reference to a charge q in reality means a line charge perpendicular to the plane having a linear charge density q.

If the value of q is restricted to the values +1 and −1 and if the subscript x is used when $q_i = +1$ and the subscript o is used when $q_i = -1$, then the formula for W becomes:

$$W = \text{const.} + \sum_{i=1}^{m} \log (z - z_{oi}) - \sum_{j=1}^{n} \log (z - z_{xj})$$

$$= \text{const.} + \sum_{i=1}^{m} \log |z - z_{oi}| - \sum_{j=1}^{n} \log |z - z_{xj}|$$

$$+ j \left(\sum_{i=1}^{m} \theta_{oi} - \sum_{j=1}^{n} \theta_{xj} \right) \quad .$$

Though the charge magnitudes have been restricted to unity, coincident charges are allowed.

The complex potential has the same mathematical form as the logarithm of a transfer function. This is the basis of the potential analogue method.

The magnitude of the logarithm of the transfer function corresponds to the real potential and the phase function to the stream function of the complex potential. Any restrictions imposed on the locations of the poles and zeros of the transfer function shall likewise be applied to the locations of the positive and negative charges, respectively, in the potential analogue.

On the basis of potential theory the potential function, or analogous transfer function, may be mapped into an auxiliary plane by means of conformal transformations. The reason for so doing is that the mapping of the original coordinates is arranged by appropriate transformations in such a fashion that one's experience and intuition in electrostatics dictates an approach for obtaining a desired transfer function.

To obtain the flat portion of a filter characteristic using the analogy, one would desire a constant potential over the corresponding portion of the real frequency axis or its mapping into another plane. A constant potential is impossible with a finite number of lumped charges and therefore it is necessary to approximate a continuous charge distribution with the lumped charges. A basic conclusion from potential theory is that a conductor enclosing a charge-free region has a constant potential on and within it. The approximation procedure then is to choose an appropriate contour, which is considered as a conductor. The charge distribution on the conductor is then calculated, if not already obvious from the choice of contour. This distributed charge is then divided into segments such that the charge of each segment is equal to that of each of the other segments. Each charge segment is then replaced by a lumped charge with the same quantity of charge. Thus in obtaining a flat pass band for a filter, a contour may be placed about the portion of the real frequency

axis corresponding to the pass band and then quantized as indicated above. This procedure requires the placing of positive charges on both sides of the $j\omega$ axis and this is prohibited by the restrictions on the transfer function pole locations.

To obviate this difficulty, encountered with the contour technique, the potential analogy will be used with respect to the gain function. The gain function is defined and used herein as a function equal to the squared magnitude of the transfer function for $p = j\omega$. If the transfer function is expressed as:

$$t(p) = K \frac{\prod_{i=1}^{m} (p - p_{oi})}{\prod_{j=1}^{n} (p - p_{xj})}$$

then $$|t(p)|^2 = K^2 \frac{\prod_{i=1}^{m} |p - p_{oi}|^2}{\prod_{j=1}^{n} |p - p_{xj}|^2}$$

If $$p_{oi} = \sigma_{oi} + j\omega_{oi} \quad \text{and} \quad p_{xj} = \sigma_{xj} + j\omega_{xj} ,$$

then $$\left|t(p)\right|^2_{p=j\omega} = K^2 \frac{\prod_{i=1}^{m} [\sigma_{oi}^2 + (\omega - \omega_{oi})^2]}{\prod_{j=1}^{n} [\sigma_{xj}^2 + (\omega - \omega_{xj})^2]} .$$

Let p'_{oi} and p'_{xj} be the negative conjugates of p_{oi} and p_{xj} respectively,

$$p'_{oi} = -\sigma_{oi} + j\omega_{oi}$$

and $$p'_{xj} = -\sigma_{xj} + j\omega_{xj} .$$

Define a quantity, G(p), as follows:

$$G(p) = K^2 \frac{\prod_{i=1}^{m} [(p - p_{oi})(-p + p'_{oi})]}{\prod_{j=1}^{n} [(p - p_{xj})(-p + p'_{xj})]}$$

$$[G(p)]_{p=j\omega} = K^2 \frac{\prod_{i=1}^{m} [\sigma_{oi}^2 + (\omega - \omega_{oi})^2]}{\prod_{j=1}^{n} [\sigma_{xj}^2 + (\omega - \omega_{xj})^2]}$$

$$= \left| t(p) \right|^2_{p=j\omega}$$

Thus it is seen that G(p) is a function satisfying the requirements of the defined gain function. It is to be noted for future use that the poles and zeros of G(p) are those of t(p) plus the mirrored images about the $j\omega$ axis. Due to this symetrical arrangement of the allowed locations of the poles of G(p), the contour technique is usable provided the contour and the negative charge (zero) locations are also symetrical to the $j\omega$ axis. For ease and simplicity the contour is chosen, when possible, such that the charge distribution is so simple and obvious that the contour itself is not directly considered in the solution of the synthesis problem.

The design problem solved in chapter III is simplified by choice of transformations to the extent that the potential distribution need never be calculated, but it is the potential analogue that provides the guidance for forming the desired gain function.

CHAPTER III

THE NEW FILTER

1. The filter type.

The problem taken up at this point is the development of a filter gain function having the following characteristics:

(a) Maximally flat in the pass band, and
(b) Equal, specified maxima in the stop band.

As is common in filter synthesis, only the low pass case will be considered since the high pass and band pass types may be obtained from the low pass circuit itself or from a transformation upon the p-plane. A filter is defined as maximally flat if its loss function, the reciprocal of its gain function, has its first n-1 derivatives, taken with respect to ω , equal to zero at some point on the real frequency axis where n is the number of poles in the transfer function. It should be noted that the requirement of equal maxima of gain in the stop band does not necessarily mean mathematical maxima, but rather, maximum values. Two exceptions can occur. In one instance the magnitude of the gain function may be approaching its maximum allowed value as ω approaches infinity. The other instance is at the defined edge of the stop band, at which frequency the gain function magnitude passes through the maximum allowed value for the stop band.

The question may be asked, "Why bother with this new filter type?" The answer lies in engineering requirements. The theory that the ear is totally insensitive to phase is being displaced. Without a reference the ear, or any other instrument, cannot detect or measure phase. Variations in relative phase delay imposed on two musical tones heard by one's ear

is undetectable. However, when transient sounds are considered, the ear, while perceiving frequency components, does not necessarily make a Fourier analysis. The timbre of even a steady sound is dependent not only on its frequency composition but also upon the phase relations of the frequency components. This factor is a frequently neglected item in the reproduction of speech and consequently intelligibility is degraded. The filter type to be subsequently developed has fairly linear phase as compared with the equal ripple pass band variety. The operation of many present day commercial and military equipments is dependent upon transient phenomenon. For example, teletype and pulse coding systems can tolerate some distortion of the pulse shapes but cannot use signals suffering from a great deal of delay distortion.

Another factor making a maximally flat filter type useful is the fact that its gain in the pass band monotonically approaches the allowed deviation from the desired constant gain. Over most of the pass band the deviation is much less than the tolerance. The overall quality of signal reproduction is better for this filter type than for the equal ripple pass band type with the same tolerance allowed.

In the region of transition between pass band and stop band, the filter herein designed has properties very nearly equivalent to those of the equal ripple pass band filter. Any filter having frequencies of infinite loss may have these used to satisfy additional requirements beyond the basic filter specifications.

For comparison purposes, the pass band characteristics of three filter types are shown in figure 1.

2. The method of solution.

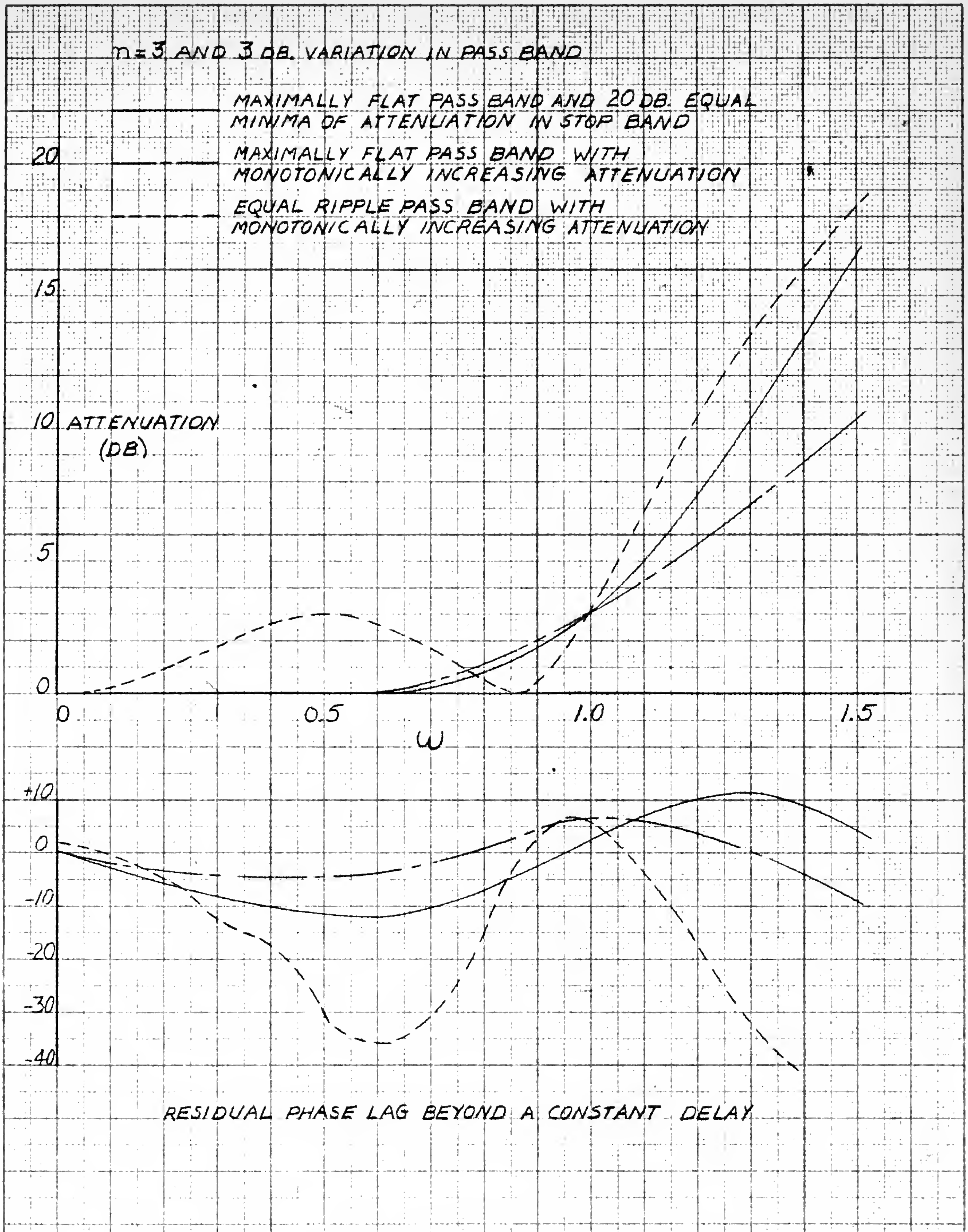


FIGURE 1. COMPARISON OF FILTER CHARACTERISTICS

The potential analogue method combined with various conformal transformations provides particularly direct, and intuitively obvious, ways of producing "equal ripple" in the stop band. "Equal ripple" with reference to the stop band portion of the gain function is a slight misnomer. It is a semantic convenience and as used herein means that the gain function has equal maximum values of magnitude in the stop band without regard to its behavior otherwise. For example, if the transformation $p = \operatorname{csch} z$ is used, the real frequency axis for $|\omega| \geq 1$ is mapped into, and includes all of, the imaginary axis in the z-plane as shown in figure 2. Then if the stop band is defined for $|\omega| \geq 1$, and the zeros are equally spaced along the imaginary axis in the z-plane, it is immediately apparent from the potential analogue that the equal ripple condition is produced provided certain other conditions are met. As previously shown, the transfer function has half the number of poles and zeros included in the gain function and the remainder must be the images as mirrored by the real frequency axis. Since the zeros are of integral order in the transfer function, any zeros of the gain function on the real frequency axis must be even ordered. The poles of the gain function must be located so as not to destroy the equal ripple condition tentatively produced with the zeros. One way of doing this is to place a pair of poles on opposite sides of the imaginary axis, equidistant therefrom, for each second order zero located on this axis. If the ordinates of the pole locations are equal to that of the corresponding zero, then the equal ripple is not destroyed. Additionally, the locations of the critical points in each cell of the z-plane must map into the same set of locations in the p-plane. This is accomplished by making the locations of those critical points in

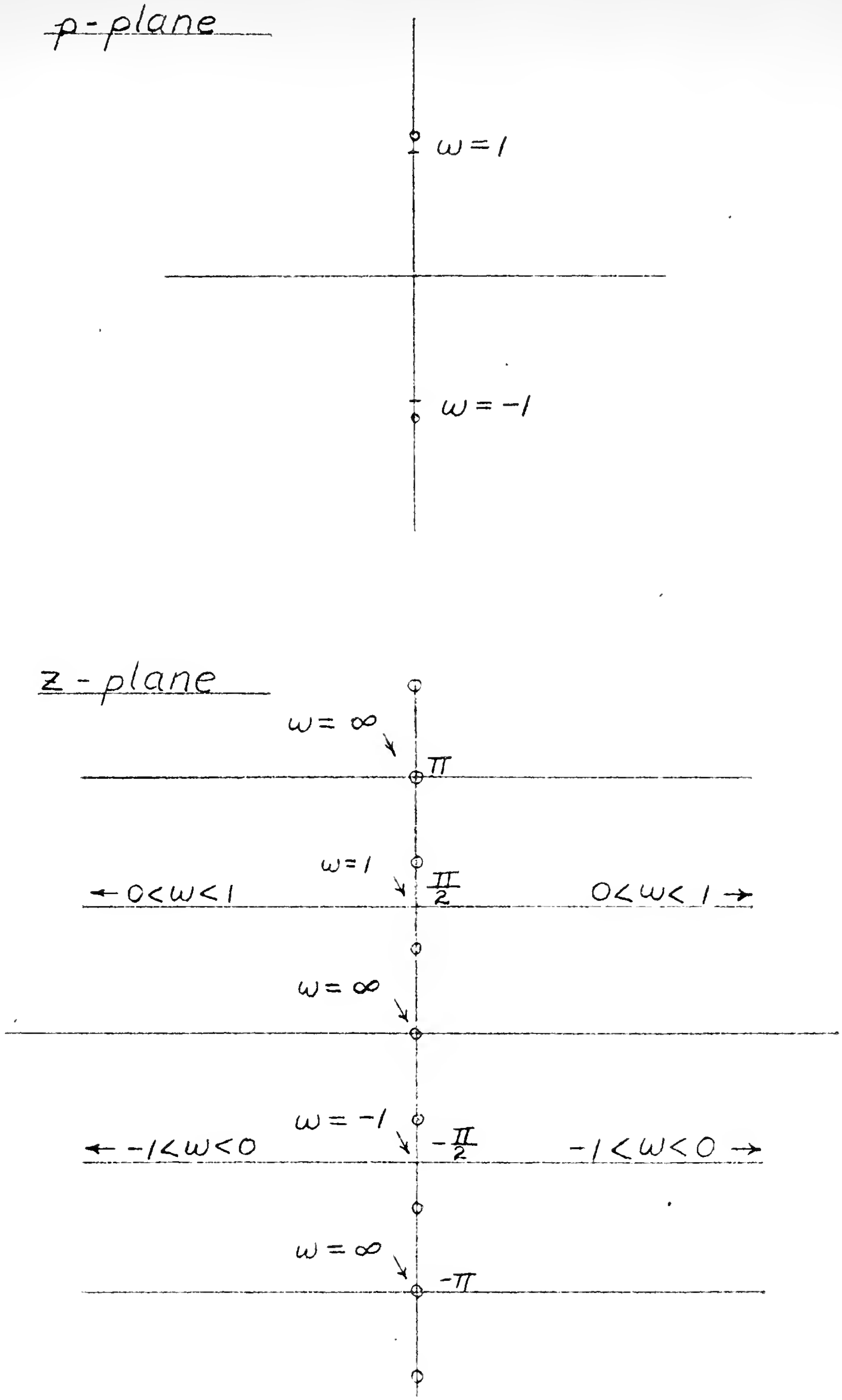


Figure 2. The transformation, $p = \operatorname{csch} z$

that cell of the z plane which includes the origin symetrical to the origin and further requiring the spacing between adjacent zeros along the imaginary axis to be $\frac{\pi}{n}$, where n is any integer. Figure 2 shows n = 3. Other considerations would apply were this transformation actually used. It has not been used since it would require the summation of an infinite series to find the gain function and a simpler means is available. Another reason will be brought out later.

Another easy method of approaching equal ripple stop band is to use a transformation such as to map the stop band portion of the real frequency axis to a circle in an auxiliary plane. The pair of transformations, $p = \sqrt{\omega^2 - 1}$ and $w = \frac{z-1}{z+1}$, does this as shown in figure 3. Again the method of producing equal ripple is apparent; that is, by equally spacing the zeros around the circle. Similar additional considerations are required as before. One advantage has been gained in that this type of transformation does not require an infinite series summation to derive the gain function.

Inherent in both the preceeding types of transformation is the same deficiency. This has to do with producing the maximally flat pass band with assurance that it actually is such. Before specifically locating this deficiency the means used to produce a maximally flat condition will be described.

The frequency at which the n-1 derivatives are made equal to zero is taken as zero, both for simplicity and to make maximally flat most meaningful if the derived characteristic is to be transformed to the band pass case. Consider a loss function as follows:

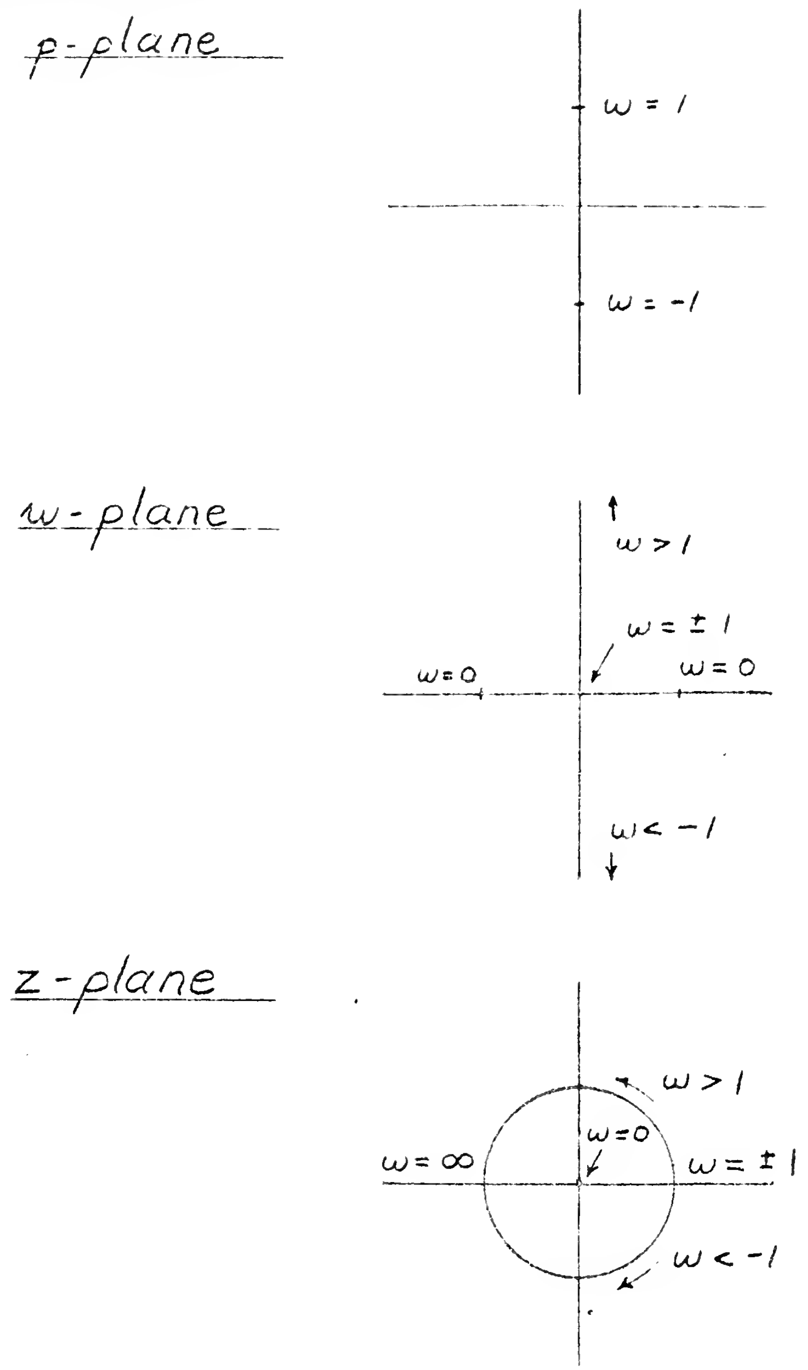


Figure 3. The pair of transformations:

$$p = \sqrt{w^2-1} \quad \text{and} \quad w = \frac{Z-1}{Z+1}$$

$$F(p) = \frac{1}{G(p)}\ .$$

Let $$F(\omega) = \frac{\omega^n}{D}\ ,$$

where D is a rational polynomial not equal to zero at $\omega = 0$.

Then, $$F'(\omega) = \frac{Dn\omega^{n-1} - \omega^n D'}{D^2}$$

$$F''(\omega) = \frac{D^3 n(n-1)\omega^{n-2} - 2D^2D'n\omega^{n-1} + [2D(D')^2 - D^2D'']\omega^n}{D^4}$$

Continuing the process of taking successive derivatives it may be seen that each term of the i^{th} derivative contains a power of ω, the lowest of which is n-i. When the n^{th} derivative is taken this is no longer true. Thus if these derivatives are evaluated at $\omega = 0$ there will be at least the first n-1 of them equal to zero. Thus the sufficient condition for $F(\omega)$ to be maximally flat is that it have an n^{th} order zero at $\omega = 0$ and necessarily it may not have a pole at the origin.

Since the derivative of a constant is zero then F + C, where C is a constant, is also maximally flat. The addition of the constant leaves one with the same pole locations but locates a new set of zeros. Remembering that the gain function is the reciprocal of the loss function, the procedure for producing the desired characteristics is to take zeros producing equal ripple in the stop band, form a maximally flat loss function with poles corresponding to the gain function zeros, add a constant, take the reciprocal of this new function as a possible gain function, and then in-

vestigate this latter function. First, the pole locations must satisfy realizability requirements. Second, the pole locations must not destroy the equal ripple character previously set up.

The additional deficiencies of the transformations shown in figures 1 and 2 will now be considered. If the p-plane is mapped into an auxiliary plane by some transformation and the $j\omega$ axis is mapped into a curve, or straight line, in this auxiliary plane, then denote distance along the curve by s. Label the derivatives of s with respect to ω as s', s'', and so on, and label the derivatives of the loss function with respect to s as F', F'', and so on. It is then found that:

$$\frac{dF}{d\omega} = s'F'$$

$$\frac{d^2F}{d\omega^2} = (s')^2F'' + s''F'$$

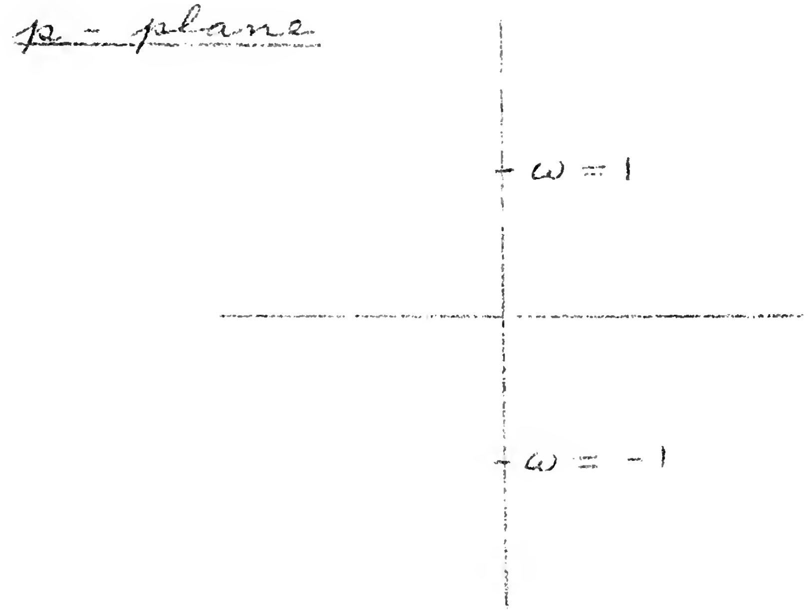
$$\frac{d^3F}{d\omega^3} = (s')^3F''' + 3s's''F'' + s'''F'$$

and so on for successive derivatives. Thus if a function is synthesized in an auxiliary plane by the procedure previously described so as to be maximally flat with respect to s; then to insure that the function, when mapped back into the p-plane, is maximally flat with respect to , it will be required that the derivatives of s with respect to ω exist when evaluated at $\omega = 0$. It is found that the previous transformations do not meet this requirement.

A transformation meeting the requirement above is $p = \frac{2z}{1-z^2}$, shown in figure 4. This one will be used to develop the gain function.

3. Development of the gain function.

First of all it should be noted that two sheets of the p-plane are



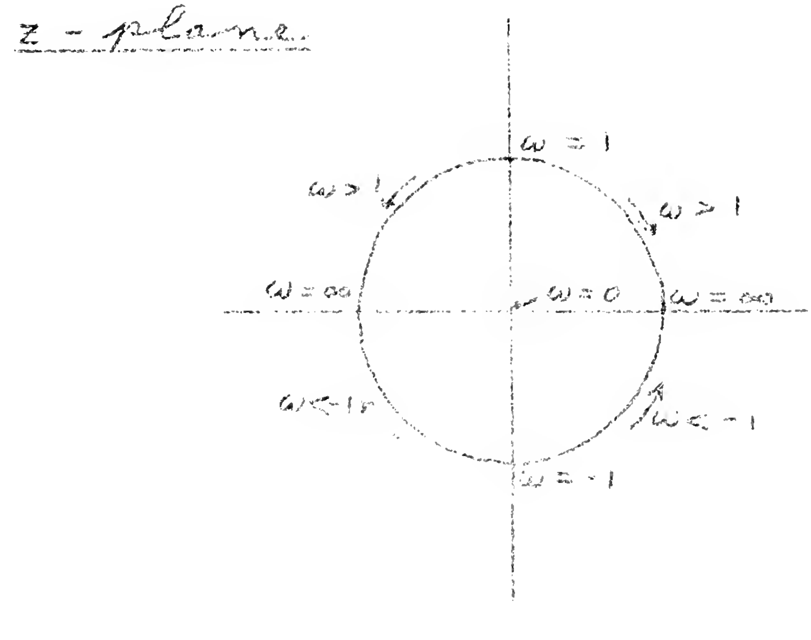


Figure 4 The transformation, $p = \frac{2z}{1 - z^2}$.

required to completely map the z-plane. One sheet maps into the interior of the unit circle in the z-plane centered at the origin, and the other to the exterior. Therefore, when placing charges (poles and zeros) in the z-plane, a pair of charges must be used for each one finally located in the p-plane, and each of a pair must map into the same coordinates on different sheets of the p-plane. If p_1 is a point in the p-plane then:

$$p_1 = \frac{2 z_1}{1 - z_1^2}$$

$$p_1 z_1^2 + 2 z_1 - p_1 = 0$$

$$z_1 = \frac{1}{p_1} \pm \sqrt{\frac{1}{p_1^2} + 1}$$

$$= \frac{1 \pm \sqrt{1 + p_1^2}}{p_1}$$

$$= \frac{1 - (1 + p_1^2)}{p_1 \left(1 \mp \sqrt{1 + p_1^2}\right)}$$

$$= \frac{-p_1}{1 \mp \sqrt{1 + p_1^2}}$$

Therefore a pair of points in the z-plane, which map into the same location in the p-plane, have complex coordinates that are negative reciprocals.

The zeros of the gain function will now be located with the requirements following as guides.

(a) n equals the number of poles in the transfer function.

(b) There shall also be n zeros in the p-plane, including the one at infinity for n odd, in order to produce the maximally flat condi-

tion and to maintain the type of symmetry required by the pole locations and the equal ripple condition.

(c) In order that the transfer function may be realized by a practical ladder network, all zeros shall be located on the $j\omega$ axis, except for a possible one at infinity, as shown by Darlington(3).

(d) In the p-plane there shall be 2n poles and 2n zeros for the gain function with mirror symmetry about both the real and imaginary axes. The zeros shall occur as doubles.

(e) There shall be 4n poles and 4n zeros in the z-plane in accordance with the mapping conditions.

(f) Since the stop band portion of the real frequency axis maps into a circle in the z-plane, the zero locations shall have equal angular spacing between them to produce the equal ripple condition.

(g) The $\omega = 1$ mapping shall be midway between adjacent zero locations. Were this point to be a zero location, the normalization of the defined edge of the stop band would not be possible.

In line with the above listed requirements the development of the gain function is started by placing second order zeros about the unit circle in the z-plane with an angular spacing of $\frac{\pi}{n}$ between successive locations. The angular displacement of the set of zeros nearest the imaginary axis from that axis is $\frac{\pi}{2n}$. This is shown in figures 5a for $n = 3$. and in figure 5b for $n = 4$. This set of zero locations satisfies all requirements placed upon them.

The maximally flat condition will now be met. A loss function is set up with poles located at the zero locations of the gain function. The maximum possible number of zeros of the loss function will be taken

at the origin.

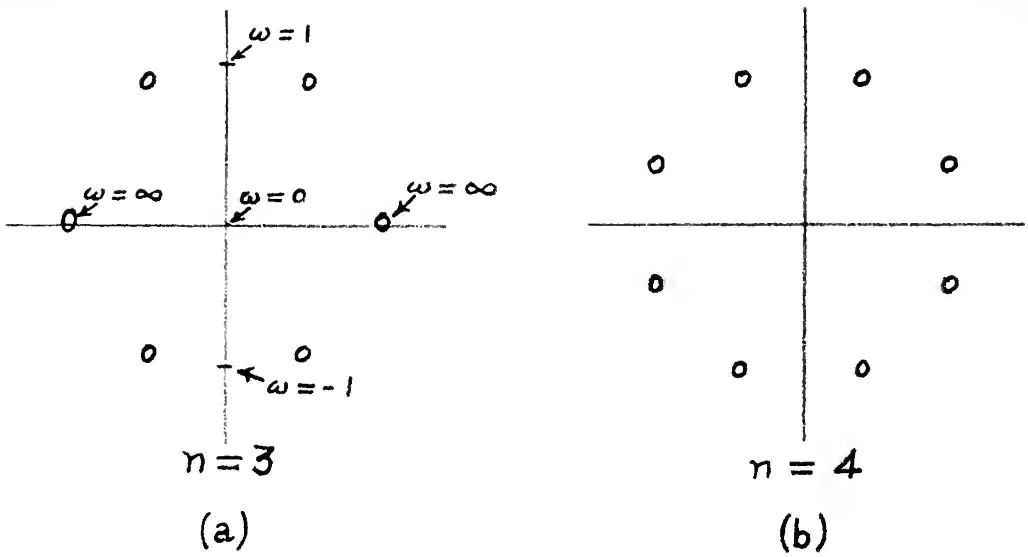


Figure 5. Zero locations in the z-plane.

There are to be 4n zeros in the z-plane and only 2n of them may be independently located. Therefore 2n zeros are placed at the origin and the other 2n zeros must be located at infinity to meet the mapping requirement. Before adding a constant, it is found to be convenient to make an additional transformation, $z^n = w$. This transformation is shown in figure 6 for representative values of n. Since each sheet of the w-plane is identical to the others, further calculations can be carried out in only one sheet of the w-plane. It is to be noted that the relative orientations of the mapping of the stop band and the critical points are the same for any n. Furthermore the multiplicity of the critical points in one sheet of the w-plane is independent of the value of n. The tentative loss function may now be written in the w-plane, neglecting any constant multiplier, as:

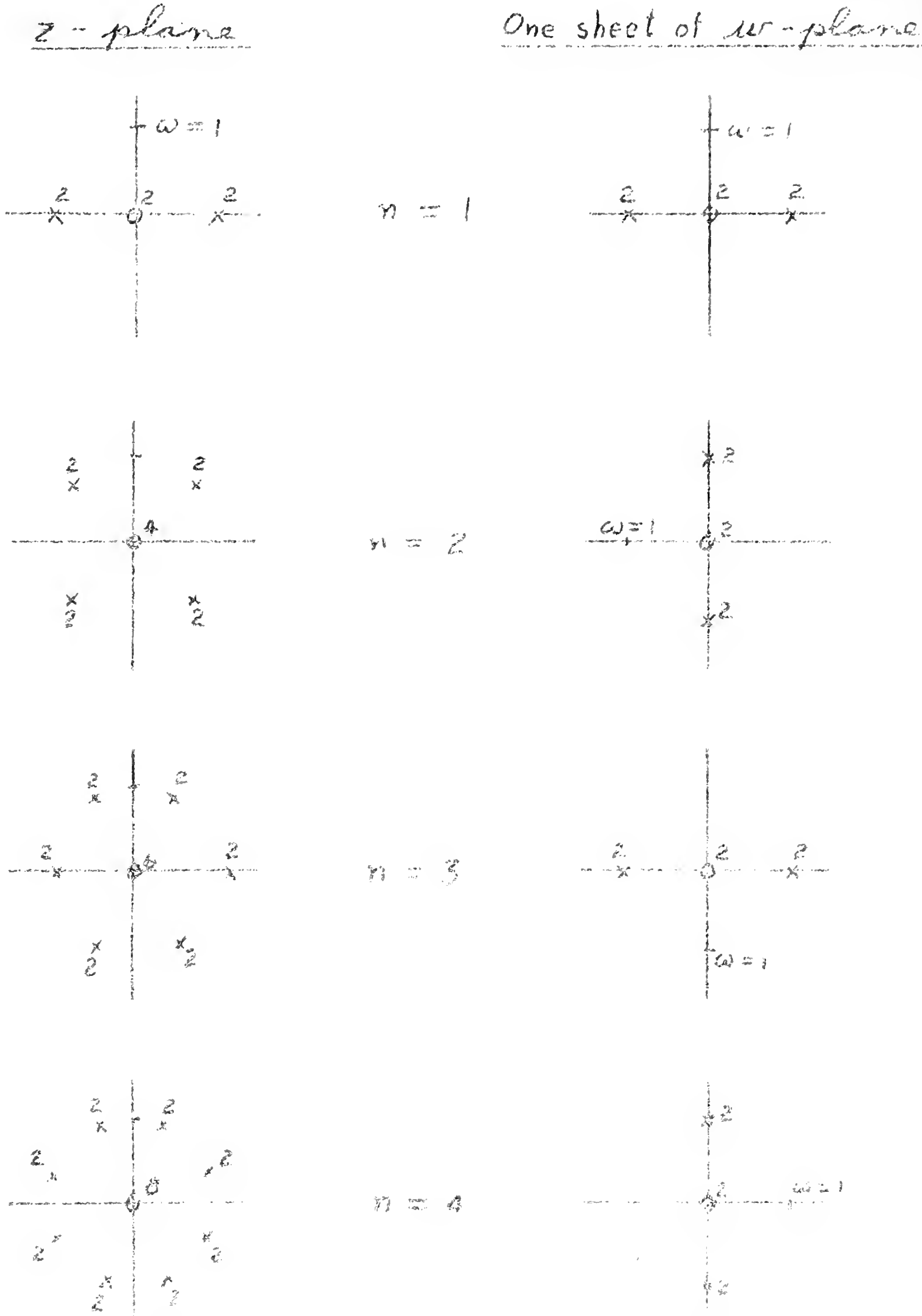


Figure 6. The transformation, $z^n = w$, for $n = 1, 2,$ 3 & 4. Critical points of F shown

$$F_1(w) = \frac{w^2}{(w-1)^2(w+1)^2} \qquad \text{for } n \text{ odd}$$

$$= \frac{w^2}{(w-j)^2(w+j)^2} \qquad \text{for } n \text{ even}$$

$$= \frac{w^2}{\left[w^2+(-1)^n\right]^2} \qquad \text{for all } n \text{ .}$$

adding a constant,

$$F_2(w) = F_1(w) + C$$

$$= \frac{Cw^4 + \left[1+2(-1)^n C\right] w^2 + C}{\left[w^2+(-1)^n\right]^2}$$

Locating the zeros of F it is found that:

$$w_0^2 = -\left[\frac{1}{2C} + (-1)^n\right] \pm \sqrt{\left[\frac{1}{2C}+(-1)^n\right]^2 - 1}$$

If C is only allowed values such that:

$C < 0$ for n odd,

and $C > 0$ for n even,

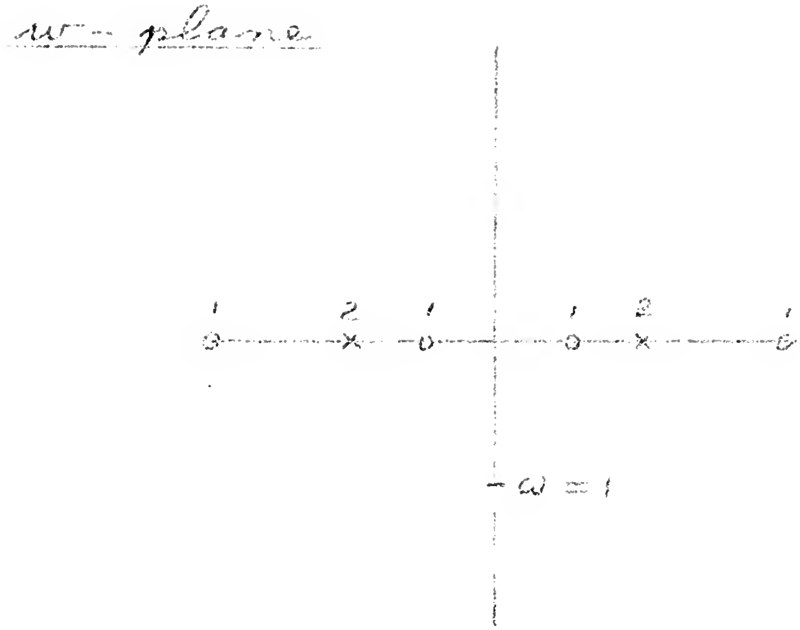
then w_0^2 is positive real for odd n and negative real for even n.

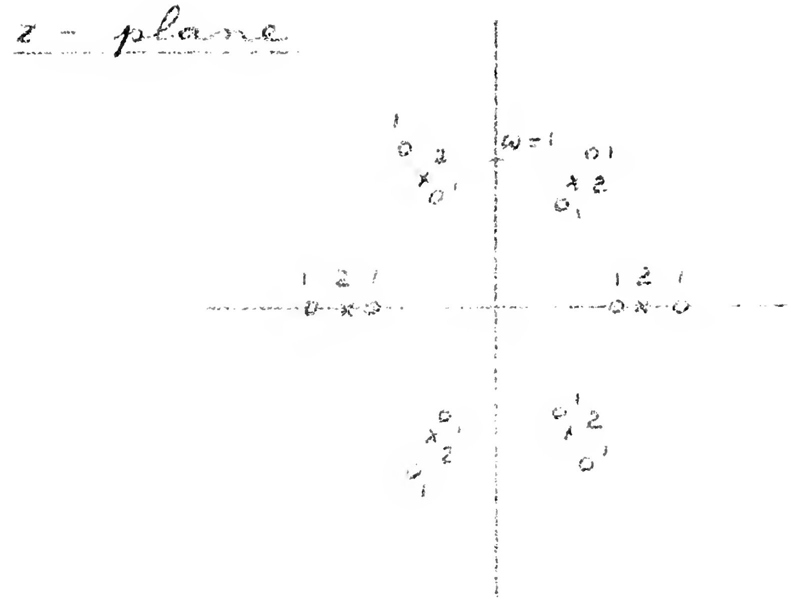
Let $$D = -\left[\frac{1}{2C} + (-1)^n\right]$$

$$w_0^2 = D \pm \sqrt{D^2-1}$$

$$\frac{D^2-(D^2-1)}{D \mp \sqrt{D^2-1}}$$

$$= \frac{1}{D \mp \sqrt{D^2-1}}$$





$n = 3$

Figure 7. Locations of critical points of F_2.

This shows that the two values of w_0^2 are reciprocals.

$$-(-1)^n w_0^{-2} = r^2, \; \frac{1}{r^2}$$

$$j^{n+1} w_0^{-2} = \pm r, \; \pm \frac{1}{r}$$

where $$r = +\left|\sqrt{D - \sqrt{D^2 - 1}}\right| .$$

The value of r has been taken arbitrarily as positive and less than one without loss of generality.

The zeros of F_2 have now been located as shown in figure 7. Since the gain function has poles at these locations, all critical points of the gain function are now known. Therefore the transfer function is also known.

At this point must be found the relations among the design specifications and the quantities which have been used in the equations. Define v to be the magnitude of w at a point which is the mapping of ω for $0 \leq \omega \leq 1$. Figure 8 shows the w-plane rotated such that the mapping of

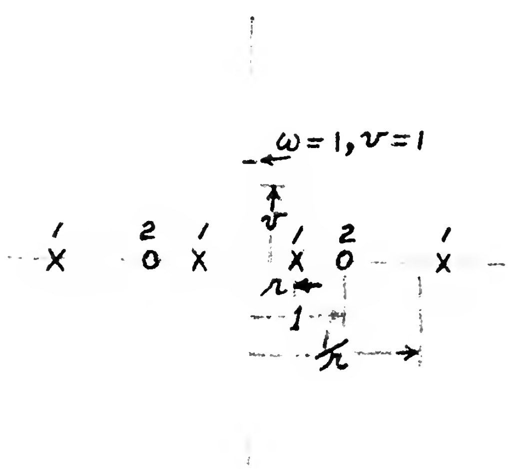


Figure 8. Critical points of G in w-plane.

the real frequencies for $0 \leq \omega \leq 1$ extends upward from the origin. The orientations of the critical points are thus independent of n. The gain function may be written from an inspection of figure 8, neglecting any constant factor, as:

$$G(v) = \frac{(v^2 + 1)^2}{(v^2 + r^2)\left(v^2 + \frac{1}{r^2}\right)}$$

Define G_s as the maximum value of G in the stop band. Since this value occurs at $\omega = 1$, then:

$$G_s = G(v)_{v=1} = \frac{4}{(1 + r^2)(1 + 1/r^2)}$$

$$G_s = \frac{4r^2}{(1 + r^2)^2}$$

$$r^2 - \frac{2}{\sqrt{G_s}} + 1 = 0$$

$$r = \frac{1}{\sqrt{G_s}} \pm \sqrt{\frac{1}{G_s} - 1}$$

As was to be expected these two values of r are reciprocals. Since r has already been chosen as the smaller value, it is most accurately computed as:

$$r = \frac{1}{\frac{1}{\sqrt{G_s}} + \sqrt{\frac{1}{G_s} - 1}}$$

r, and consequently $G(v)$, is completely determined by G_s. By the omission of a constant multiplier from the gain expression such a multiplier

has been tacitly assumed as equal to unity. Thus $G(v)_{v=0} = 1$, and this is suitable for realization purposes.

Define G_p as the minimum allowed value of G in the pass band and k as that value of ω which is the upper limit of the pass band. Since $\omega = 1$ defined the lower limit of the stop band, then $0 < k < 1$.

Take $\quad p = \dfrac{2z}{1 - z^2}$, where $\quad z = x + jy.$

Then $\quad pz^2 + 2z - p = 0$

$$z = -\frac{1}{p} \pm \sqrt{\frac{1}{p^2} + 1}$$

Since these two values of z are negative reciprocals for all values of p, only one need be considered and the other is an automatic consequence. Using the expression,

$$z = -\frac{1}{p} \pm \sqrt{\frac{1}{p^2} + 1}$$

$$[z]_{p=j\omega} = \frac{j}{\omega} - \sqrt{-\left(\frac{1}{\omega^2} - 1\right)}$$

$$[x + jy]_{p=j\omega} = j\left[\frac{1}{\omega} - \sqrt{\frac{1}{\omega^2} - 1}\,\right]$$

$$\therefore \; x = 0 \quad \text{and} \quad y = \frac{1}{\omega} - \sqrt{\frac{1}{\omega^2} - 1}$$

$$\omega = \frac{2y}{1 + y^2}$$

From $z^n = w$ it is seen that: $y^n = v$.

Take the equation for G(v) and write its reciprocal:

$$\frac{1}{G(v)} = \frac{(v^2 + a^2)\left(v^2 + \frac{1}{a^2}\right)}{(1 + v^2)^2}$$

$$\frac{1}{G(v)} = \frac{1 + v^4 + 2v^2 - 2v^2 + v^2(r^2 + 1/r^2)}{(1+v^2)^2}$$

$$= 1 + \frac{v^2(r^2 - 2 + 1/r^2)}{(1+v^2)^2}$$

Substituting, $$r = \frac{1}{\sqrt{G_s}} - \sqrt{\frac{1}{G_s} - 1}$$

and, $$\frac{1}{r} = \frac{1}{\sqrt{G_s}} + \sqrt{\frac{1}{G_s} - 1}$$

Then $$\frac{1}{G(v)} = 1 + \frac{v^2}{(1+v^2)^2}\left[\left(\frac{1}{\sqrt{G_s}} - \sqrt{\frac{1}{G_s} - 1}\right)^2 - 2 + \left(\frac{1}{\sqrt{G_s}} + \sqrt{\frac{1}{G_s} - 1}\right)^2\right]$$

$$= 1 + \frac{4v^2}{(1+v^2)^2}\left(\frac{1}{G_s} - 1\right)$$

$$\left(\frac{1}{G(v)} - 1\right) = \left(\frac{1}{G_s} - 1\right)\frac{4v^2}{(1+v^2)^2}$$

Let $$Q(v) = \frac{\frac{1}{G(v)} - 1}{\frac{1}{G_s} - 1}$$

$$Q(v)\,(1+v^2)^2 = 4v^2$$

$$Q(v)\,v^4 + 2\left(Q(v) - 2\right)v^2 + Q(v) = 0$$

$$v^4 + 2\left(1 - \frac{2}{Q(v)}\right)v^2 + 1 = 0$$

$$v^2 = S \pm \sqrt{S^2 - 1}$$

where $$S = \frac{2}{Q(v)} - 1$$

Of the two reciprocal values of v^2 given by the $\pm$ sign above, only that one associated with the minus sign will be considered.

If $$S_k = \frac{2}{Q_k} - 1$$

where $$Q_k = Q(v_k)$$

and v_k is that value of v for $\omega = k$, then since $v^2 = y^{2n}$

$$S_k - \sqrt{S_k^2 - 1} = \left[\left(\frac{1}{\omega} - \sqrt{\frac{1}{\omega^2} - 1} \right)^{2n} \right]_{\omega = k}$$

$$= \left(\frac{1}{k} - \sqrt{\frac{1}{k^2} - 1} \right)^{2n}$$

$$n = \frac{\log \left(S_k - \sqrt{S_k^2 - 1} \right)}{2 \log \left(\frac{1}{k} - \sqrt{\frac{1}{k^2} - 1} \right)}$$

Since n must be an integer, the specification of G_s, G_p and k cannot be completely arbitrary. The normal procedure is to specify G_s, G_p, and the minimum value of k. S_k is computed from G_s and G_p. Then n is taken as the least integer equal to or larger than the value computed from the above

equation.

4. A design nomograph.

For a practicing engineer interested in choosing a suitable filter for a particular application, the calculations required by the preceeding formulae are lengthy and tedious. To ease this situation a nomograph is developed to enable one to rapidly choose the value of n required to meet the basic filter specifications and then easily derive the corresponding gain curve. A summary of pertinent formulae follows:

$$\omega = \frac{2y}{1+y^2} \quad \text{for } 0 \leq \omega \leq 1$$

$$y^n = v$$

$$\frac{4v^{-2}}{(1+v^{-2})^2} = Q$$

$$Q = \frac{\frac{1}{G} - 1}{\frac{1}{G_s} - 1}$$

Exponents are prominent in the first three of these equations and they are plotted on logarithmic graph paper as the basic nomograph. The fourth equation is plotted separately and this plot is used to provide easy entry into the nomograph. These plots are given in appendix A and B respectively, and a sketch of the basic nomograph is shown in figure 9.

The filter designer may also be interested in the frequencies of infinite loss and those of maximum stop band gain. These are tabulated in Appendix C for various values of n.

It may be noted that neither the nomograph in Appendix A nor the tabulation in appendix C includes data for n = 1. For n = 1 this filter

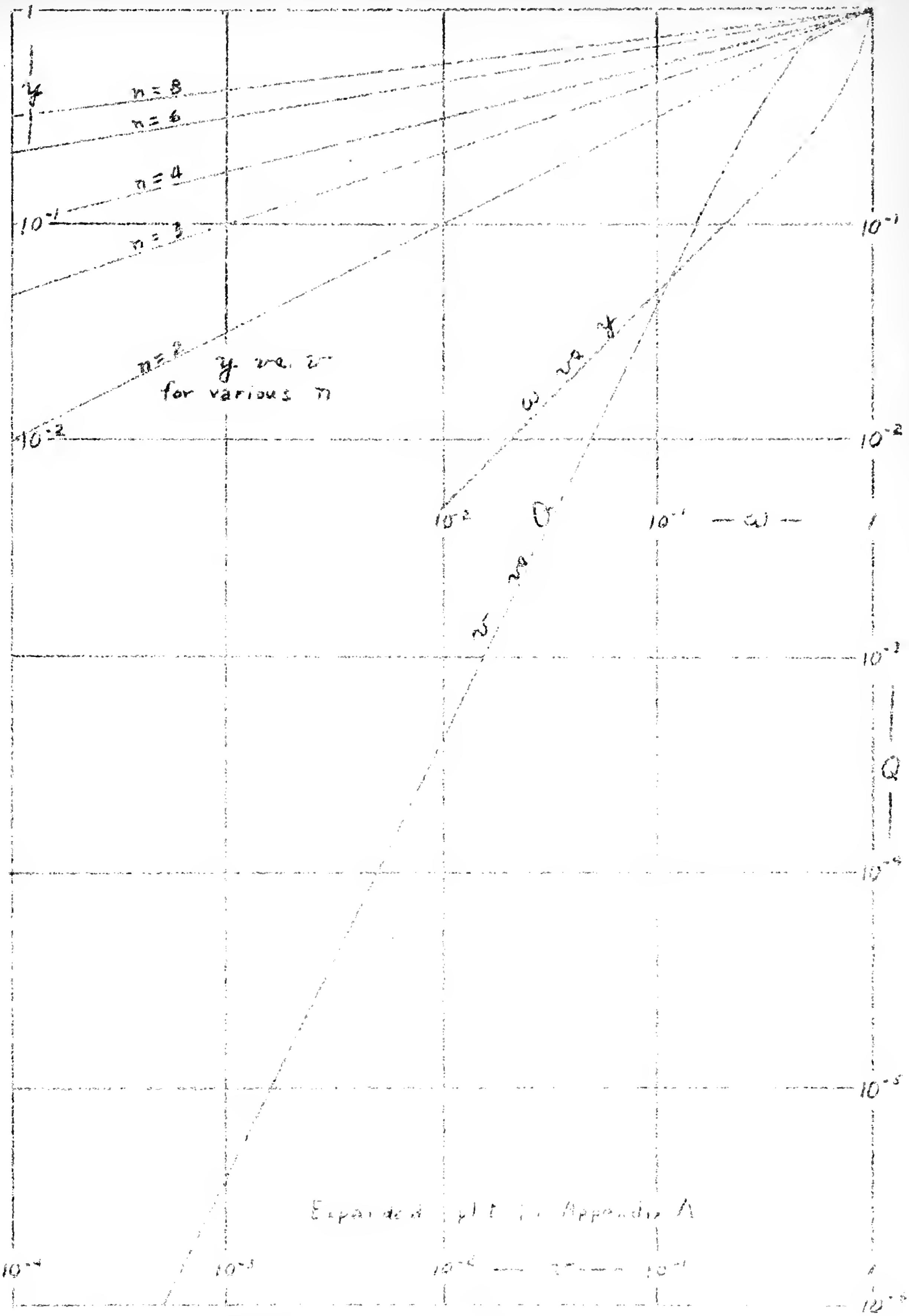


Figure 7 Design Nomograph

type is identical to the Butterworth and the equal ripple pass band types.

5. Realizing the filter network.

Before finding the physical configuration of the filter, it will be necessary to find the transfer function in terms of p. To do this, the locations of the poles and zeros in the w-plane must be mapped back into the p-plane. The quantity r is found from G_s as shown on page 28, and n is found from the nomograph. Then in the z-plane the poles of the gain function lie on circles centered at the origin and with radii, R and 1/R, where $R = \sqrt[n]{r}$. Only those on the inner circle need be considered and their angular locations are the same as those of the zeros, as described in section 3. These pole locations are then mapped into the p-plane. Only those in the left half-plane are used as explained in section 2 of chapter II. The zeros of the transfer function are simple and lie on the $j\omega$ axis. The values of ω at these locations, ω_o , are given in appendix C as the frequencies of infinite loss. Zeros are located at $p = +j\omega_o$ and $p = -j\omega_o$ for each value of ω_o given in appendix C. No action need be taken for $\omega_o = \infty$ since a pole occurs naturally there whenever n is odd. The transfer function is now written in the usual form with m = n or n − 1, whichever is even.

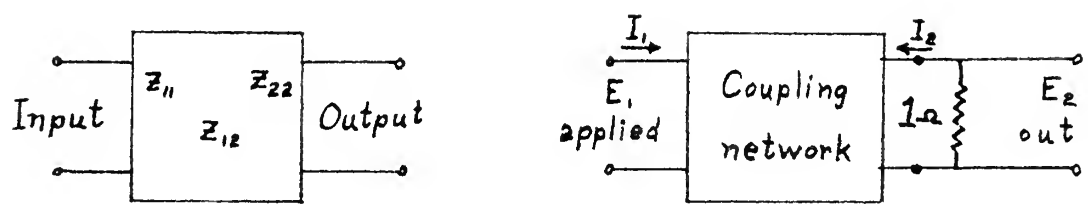
$$t(p) = K \frac{\prod_{i=1}^{m} (p - p_{oi})}{\prod_{j=1}^{n} (p - p_{\kappa j})}$$

$$t(p) = K \frac{p^m + a_{m-2} p^{m-2} + a_{m-4} p^{m-4} + \cdots + a_2 p^2 + a_0}{p^n + b_{n-1} p^{n-1} + b_{n-2} p^{n-2} + \cdots + b_1 p + b_0}$$

Only even powers of p occur in the numerator since the zeros are located at conjugate values on the imaginary axis. K is chosen as b_0/a_0 so that $t(p) = 1$ at $p = 0$.

With the transfer function now in hand the network is arrived at by a procedure based on a method originally given by Norton(7). The form of realization is that of a purely reactive coupling network terminated in a one ohm resistance. The output is taken across this resistor. The input may be either a voltage feed or current feed depending on the circuitry in the coupling network. The procedure for determining the coupling network will be derived for a voltage feed and the dual of the circuit may be used if a current feed is desired.

Consider a two terminal-pair reactive network as indicated in figure 10. z_{11} and z_{22} are the open circuit



Coupling network   Filter arrangement

Figure 10. Representation of the filter realization.

impedances of the input and output respectively. z_{12} is the transfer impedance. Using the generalized form of impedances, then z_{11}, z_{22}, and z_{12} are all odd functions of p since they are made up of purely reactive terms. Take the equations:

$$E_1 = I_1 z_{11} + I_2 z_{12} \tag{1}$$

$$E_2 = I_1 z_{12} + I_2 z_{22} \tag{2}$$

Substitute $\quad I_2 = -E_2 \quad$ (numerically) $\quad$ (3)

$$E_1 = I_1 z_{11} - E_2 z_{12} \tag{4}$$

$$E_2(1 + z_{22}) = I_1 z_{12} \tag{5}$$

Solve (5) for: $\quad I_1 = E_2 \dfrac{1 + z_{22}}{z_{12}} \quad$ (6)

Substitute (6) into (4) $\quad E_1 = E_2\left[\left(\dfrac{1+z_{22}}{z_{12}}\right) z_{11} - z_{12}\right] \quad$ (7)

$$\frac{E_1}{E_2} = \left[\frac{z_{11}}{z_{12}}\right] + \left[\frac{z_{11} z_{22}}{z_{12}} - z_{12}\right] = A + pB \tag{8}$$

where A and B are even functions of p.

$$\frac{pB}{A} = z_{22} - \frac{z_{12}^2}{z_{11}} \tag{9}$$

Define Z_2 as the impedance seen looking into the output terminals of the coupling network with the input shorted. From equation (1) with $E_1 = 0$:

$$I_1 = -I_2 \frac{z_{12}}{z_{11}} \tag{10}$$

Substituting (10) into (2): $\quad E_2 = I_2\left(z_{22} - \dfrac{z_{12}^2}{z_{11}}\right) \quad$ (11)

$$Z_2 = -\frac{E_2}{I_2} = z_{22} - \frac{z_{12}^2}{z_{11}} \tag{12}$$

From (9) and (12) $\quad Z_2 = \dfrac{pB}{A} \quad$ (13)

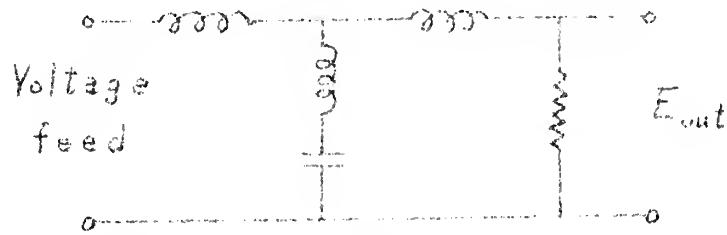
The transfer function came out to be of the form $\dfrac{C}{J + pL}$, where C, J

and L are even functions of p. Then the reciprocal is:

$$\frac{E_1}{E_2} = \frac{J + pL}{C} = A + pB \tag{14}$$

$$\frac{pB}{A} = \frac{pL}{J} \tag{15}$$

Equation (13) shows that the ratio of the odd part of the reciprocal of the transfer function to the even part is equal to the impedance seen looking into the output terminals of the coupling network when the input terminals are shorted. However, equation (15) shows that this ratio contains no information relative to the zeros of the transfer function. The coupling network is realized by synthesizing Z_2 as a driving point impedance in a ladder form. Then by opening the circuit between the final branch element and ground, consider the terminals thus produced as the input terminals. In addition, the resultant network with a one ohm load resistor must be forced to have the zeros of its transfer function at the proper locations. The procedure for accomplishing this is given by Guillemin(6). Basically, it consists of removing a branch impedance from Z_2 such that the remainder has a pair of zeros or poles located at $p = \pm j\omega_0$. A pair of zeros is removed as a resonant circuit in a shunt branch or a pair of poles as an anti-resonant circuit in a series branch of the ladder. Figure 11 shows two forms the filter may take for n = 3.



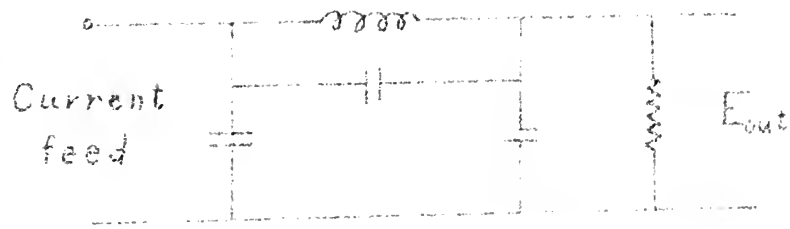


Figure 11. Filter component arrangements for n = 3.

CHAPTER IV

CONCLUSION

The basic design procedure for a new type of filter has been developed. The gain function has equal maximum values in the stop band and is described as having an equal ripple stop band characteristic. The reciprocal of the gain function, the loss function, has its first n-1 derivatives equal to zero at $\omega = 0$, and this is described as a maximally flat pass band characteristic. It can be shown that the gain function also has its first n-1 derivatives equal to zero at $\omega = 0$. The gain function which was synthesized in chapter III could have been obtained without considering the loss function, but such a procedure would not have as clearly separated the stop band and pass band problems.

A design nomograph was developed which permits a designer to determine immediately the value of n required to meet the filter specifications. Having determined n, the nomograph may be used to obtain easily a plot of the gain function without the necessity of first synthesizing the gain function.

A procedure was given for realizing the filter with a physical configuration suitable for use in vacuum tube circuits. Other realizations are, of course, possible.

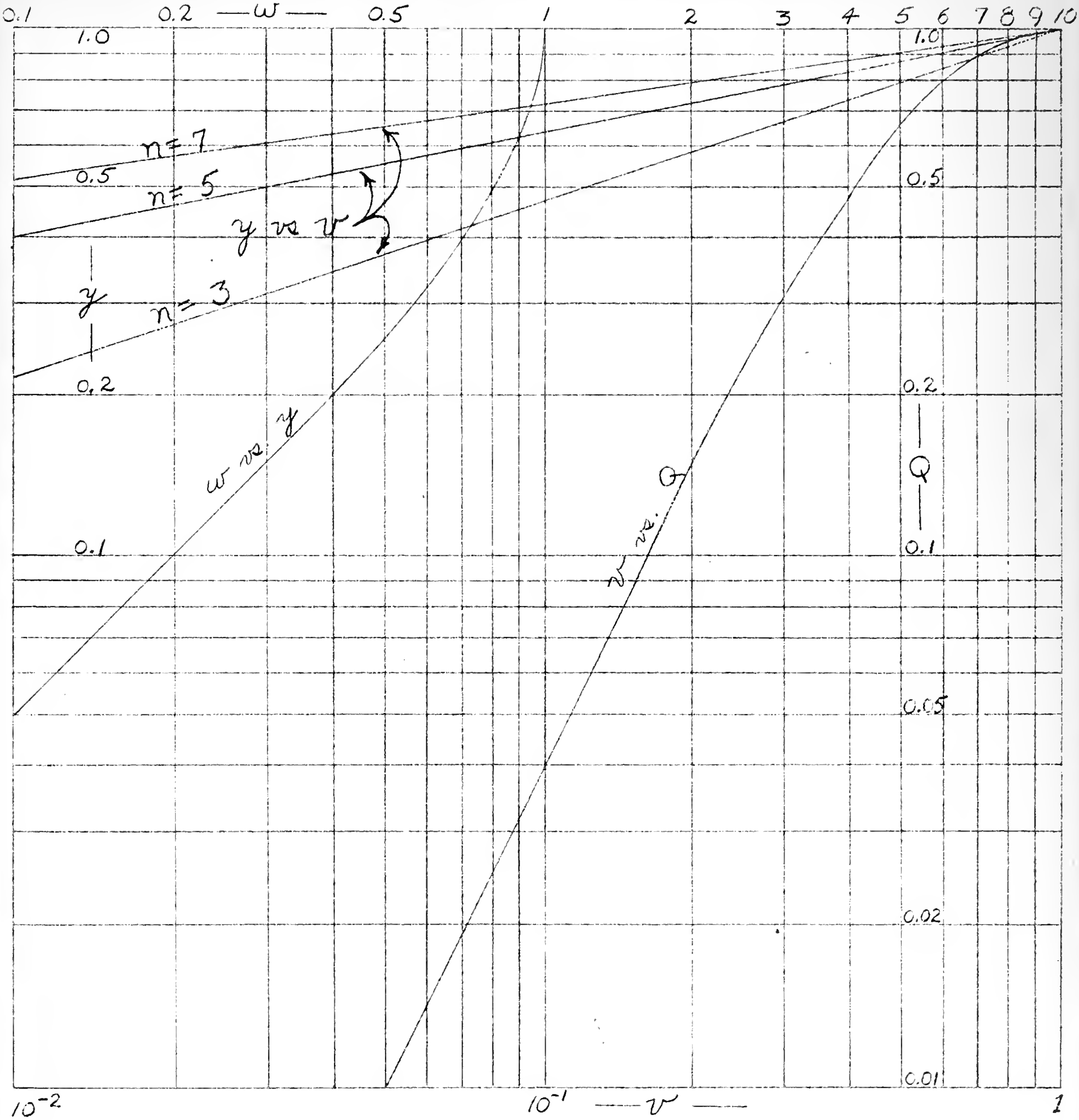
The definition of maximally flat which has been used is rather arbitrary. For a general gain function of degree 2n, there are 2n-1 independent conditions which may be imposed on the function. In the well known Butterworth function, all these conditions have been used in the specification of maximal flatness. A more general definition of maximal flatness

might be taken as:

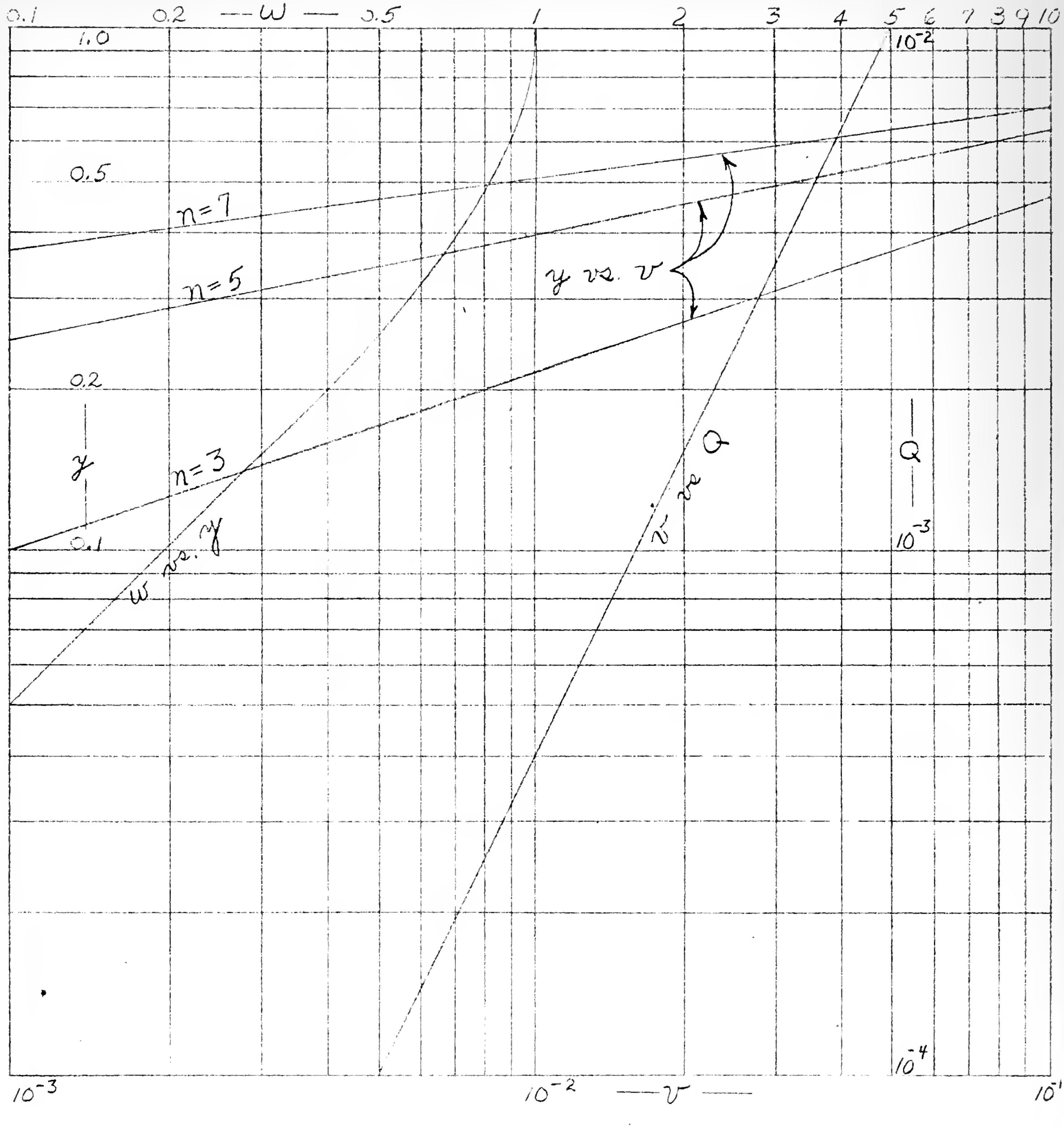
> A gain function of degree 2n is defined as maximally flat if its first 2n-1-m derivatives are equal to zero at some one frequency, where m is the number of constraints imposed on the function in the production of other desired characteristics.

Further investigations suggested are the synthesis of maximally flat gain functions of different relative degrees of flatness and those with stop band characteristics other than equal ripple.

APPENDIX A-1

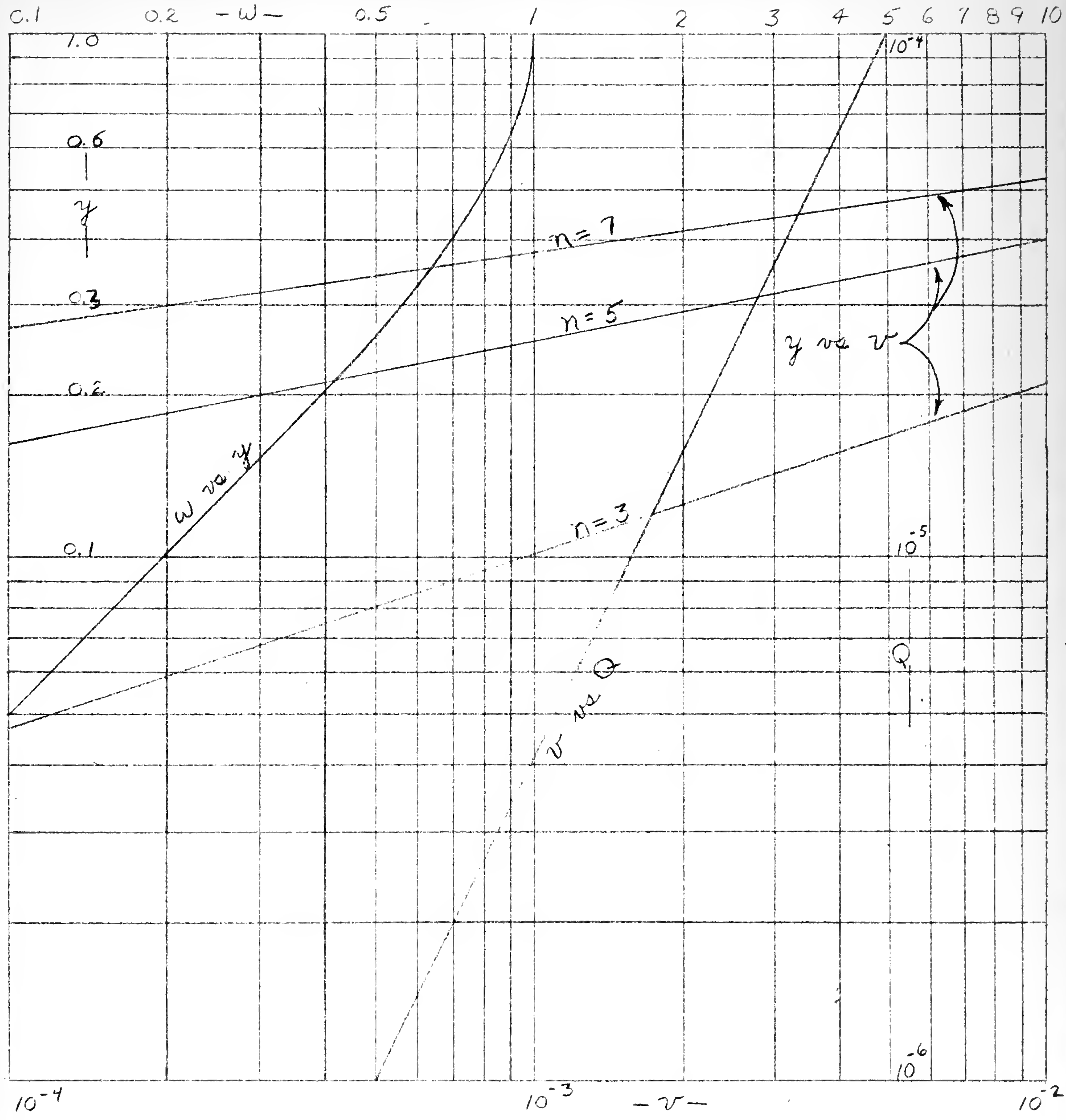

0.1
0.2
— w —
0.5
1
2
3
4
5
6
7
8
9
10
1.0
0.5
0.2
0.1
0.05
0.02
0.01
y
Q
n = 7
n = 5
n = 3
y vs v
w vs. y
v vs. Q
10^-2
10^-1
— v —
1

APPENDIX A-2



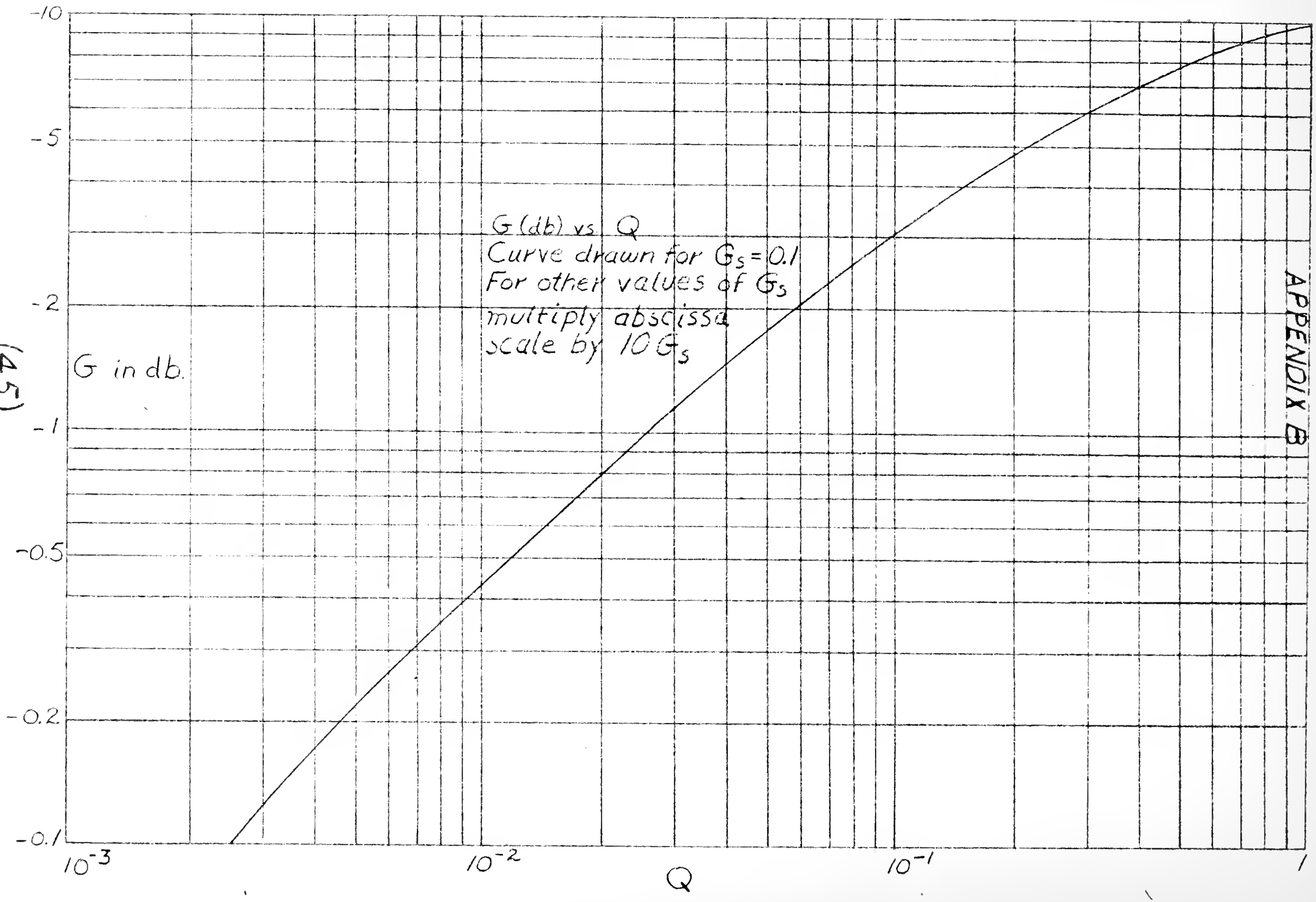
0.1
0.2
— ω —
0.5
1
2
3
4
5
6
7
8
9
10
1.0
0.5
0.2
0.1
y
n=7
n=5
n=3
y vs. v
ω vs. y
v vs Q
Q
10^{-2}
10^{-3}
10^{-4}
10^{-3}
10^{-2}
— v —
10^{-1}

APPENDIX A-3



0.1
0.2
- ω -
0.5
1
2
3
4
5
6
7
8
9
10
1.0
0.6
y
0.3
0.2
0.1
10^{-4}
10^{-5}
10^{-6}
Q
n = 7
n = 5
n = 3
ω vs y
y vs v
v vs Q
10^{-4}
10^{-3}
- v -
10^{-2}

APPENDIX B



G (db) vs Q
Curve drawn for $G_s = 0.1$
For other values of G_s
multiply abscissa
scale by $10\,G_s$
G in db.
-10
-5
-2
-1
-0.5
-0.2
-0.1
10^{-3}
10^{-2}
10^{-1}
1
Q

APPENDIX C

Tabulated frequencies of infinite loss and maximum stop band gain.

ω_o = frequency of infinite loss
ω_s = frequency at which $G(\omega_s) = G_s$

n	ω_o	ω_s
2	1.4142	1.0000
		∞
3	1.15470	1.0000
	∞	2.0000
4	1.0824	1.0000
	2.6131	1.4142
		∞
5	1.0515	1.0000
	1.7013	1.2361
	∞	3.2361
6	1.0353	1.0000
	1.4142	1.1547
	3.8637	2.0000
		∞
7	1.0257	1.0000
	1.2790	1.1099
	2.3048	1.6039
	∞	4.4939
8	1.0196	1.0000
	1.2027	1.0824
	1.8000	1.4142
	5.1258	2.6131
		∞

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